High-order level-spacing distributions for mixed systems

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(Received 4 June 1999; revised manuscript received 2 June 2000)

We apply some of the methods that have been successfully used to describe the nearest-neighbor-spacing distributions of levels of systems with mixed regular-chaotic dynamics to the calculation of high-order spacing distributions. The distributions for chaotic spectra are described in terms of a previously suggested generalization of Wigner's surmise, which assumes that the high-order level repulsion function is given by a product of the zero-order ones and that all of the spacing distributions are nearly Gaussian functions at large spacings. We compare the expressions obtained by the different methods for the next-nearest-neighbor spacing distribution with the outcome of a recently published numerical experiment on systems in transition between order and chaos. We show that the evolution of the shape of that distribution during the transition of the system from a chaotic to a regular regime is slower than the corresponding transition for the nearest-neighbor spacing distribution.

PACS number(s): 05.45.-a

I. INTRODUCTION

The random matrix theory [1,2] is a natural framework for describing the levels of quantum systems whose classical counterparts have chaotic dynamics. The theory suggests that the spacing distribution of levels of a chaotic system has a universal character that depends only on the symmetry properties of the system when the spectrum is renormalized to make the mean spacing equal to 1. For example, the nearest-neighbor-spacing (NNS) distribution of the levels of a chaotic system behaves at small spacing as $P(s)\propto s$ when the system is represented by a Gaussian orthogonal ensemble (GOE). Because in these cases the Hamiltonian matrix elements are assumed to have a Gaussian distribution, one expects the large-s behavior of the NNS distribution to be also given by a Gaussian function. The two asymptotic conditions are satisfied by the so-called Wigner surmise,

$$P_W(s)=\frac{\pi}{2}se^{-\pi s^2/4}. \qquad (1)$$

On the other hand, the NNS distribution for regular systems is in most cases given by a Poisson distribution:

$$P_R(s)=e^{-s}. \qquad (2)$$

Systems whose classical dynamics is intermediate between regularity and chaos have also been considered. Robnik [3] was probably the first to show a continuous transition in a deformed billiard from the Poisson to the Wigner distribution. Several expressions that interpolate between the two distributions have been proposed [4–12]. Some of them will be discussed below.

In a previous paper [13], we considered the nth-order spacing distribution $p(n,s)$ defined as the probability density that the distance s between two levels contains exactly n levels. In this notation, $P(s)=p(0,s)$. We applied a statistical approach suggested by Engel *et al.* [14] as a generalization of a well-known work by Wigner [15], which expresses the distributions $p(n,s)$ in terms of the nth-order level-repulsion function $r(n,s)$. They obtained a generalization of Brody's formula for the level spacing distribution of mixed systems by taking $r(n,s)\propto s^{q_n}$, and taking q_n as a free parameter. For a GOE, where $r(0,s)\propto s$, this approach yields Wigner's formula for the NNS distribution. In Ref. [13], we apply this approach to evaluate the nth-order spacing distributions $p(n,s)$ at small spacings by assuming that

$$r(n,s)\propto[r(0,s)]^{n+1}. \qquad (3)$$

The large-spacing dependence of the distributions is assumed to be dominated by Gaussian functions. The overall dependence on s of the distributions obtained by combining the two limits is found to agree with the exact results obtained in Mehta's book [1] for the gap functions $E(n,s)$ when $n=0-7$. In particular, the expressions obtained for the next-nearest-neighbor-spacing (NNNS) and the second-nearest-neighbor-spacing (second-NNS) distributions are [13]

$$p_W(1,s)=A_1s^4e^{-B_1s^2},\quad A_1=2B_1^3,\quad B_1=16/(9\pi), \qquad (4)$$

and

$$p_W(2,s)=A_2s^8e^{-B_2s^2},\quad A_2=B_2^5/4,\quad B_2=2^{14}/(7^25^23^2\pi). \qquad (5)$$

To see this, we expand expression (4) for $p_W(1,s)$ in powers of s and keep the leading two terms to obtain $p_W(1,s)=0.362\,41s^4-0.205\,08s^6+\cdots$, which agrees reasonably well with the power-series expansion $p_W(1,s)=0.360\,77s^4-0.230\,47s^6+\cdots$ obtained in [1]. Similarly, the leading term in the power-series expansion of expression (5) for $p_W(2,s)$, $5.9208\times10^{-3}s^8$, is consistent with the value $5.3790\times10^{-3}s^8$ obtained in [1] for the corresponding term.

The purpose of the present paper is to find expressions for the nth-order spacing distributions for mixed systems that interpolate between the corresponding distributions for the regular systems, which have Poissonian forms,

$$p_P(n,s)=\frac{s^n}{n!}e^{-s}, \qquad (6)$$

and for the chaotic systems that are described by a GOE, which are obtained in Ref. [13]. In Sec. II, we apply assumption (3) to the generalization of Brody's formula proposed in the work of Engel *et al.* [14]. Section III considers the case in which the spectrum can be described as a random superposition of a number of level sequences. We obtain explicit expressions for the NNNS distribution when the constituting sequences have equal densities and satisfy the GOE statistics. This is the case for a chaotic system in which a symmetry is unknown or ignored. We demonstrate the gradual transition of the shapes of these distributions towards the Poissonian shape as the number of constituting sequences increases. This formalism is used in Sec. IV to obtain a generalization of the Berry-Robnik formula for the high-order level spacing distributions of mixed systems. The summary and conclusion of the present work are given in Sec. V.

II. THE STATISTICAL METHOD

Wigner [15] derived an interesting formula for the NNS distribution using simple statistical arguments. The only input is the level-repulsion function $r(s)$, defined as the conditional probability that, given a level at energy E, there is one level in the interval ds at a distance s provided that there are no levels in the interval $[E,E+s]$. In terms of this function, the NNS distribution $P(s)$ has been expressed as

$$P(s)=r(s)\exp\left[-\int_0^s r(x)dx\right]. \qquad (7)$$

The Wigner distribution (1) is obtained from Eq. (7) by substituting $r(s)\propto s$ and the Poisson distribution (2) by setting $r(s)=1$. Brody [4] interpolates between the Wigner and Poisson distributions by taking $r(s)\propto s^q$, where $0<q<1$. He then uses Eq. (7) to obtain

$$p_B(0,s)=as^q\exp(-bs^{q+1}), \quad a=(q+1)b,$$

$$b=\{\Gamma[(q+2)/(q+1)]\}^{q+1}, \qquad (8)$$

where $\Gamma[x]$ is the gamma function. Brody's formula (8) has been very successful in reproducing the NNS distributions for various systems with mixed regular-chaotic dynamics. Theoretical arguments explaining the fractional power-law level repulsion are given by Prosen and Robnik [10]. The free parameter q serves as a purely phenomenological measure for the degree of mixing between Poisson and GOE statistics. Engel *et al.* [14] pursued an analogous statistical approach to determine the level-spacing distributions of the nth neighbors for mixed systems. Their derivation is based on defining $r(n,s)ds$ as the conditional probability that a new $[(n+1)$th$]$ level occurs in an interval ds at a distance s from an arbitrarily chosen level provided that this distance contains exactly n levels. They obtained

TABLE I. Mean values $\overline{q}$ and standard deviations σ_q of the chaoticity corresponding to the best-fit parameters of nth-order spacing distributions [14].

System	Hydrogen atom		Hénon-Heiles potential		
ϵ	−0.10	−0.35	0.166	0.133	0.108
$\overline{q}$	0.77	0.22	0.55	0.27	0.091
σ_q	0.09	0.002	0.09	0.01	0.009

$$p(n,s)=r(n,s)\exp[-R(n,s)]\times\int_0^s p(n-1,x)\exp[R(n,x)]dx \quad \text{for} \quad n\geq 1, \qquad (9)$$

where $R(n,s)=\int_0^s r(n,x)dx$. They applied Eq. (9) to obtain a Brody-like formula for $p(n,s)$ by proposing a power-law ansatz for the nth-order level-repulsion function:

$$r(n,s)\propto s^{q_n}, \qquad (10)$$

and considering q_n as a free parameter for each value of n.

We shall now assume that the nth-order level-repulsion function is related to the zeroth-order level-repulsion function by means of the ansatz (3). Therefore, we shall not consider the exponents q_n as free parameters as done in [14], but shall consider them to be related by

$$q_n=(n+1)q, \qquad (11)$$

where $q(=q_0)$ is the parameter used in Eq. (8) to define the degree of chaoticity of the system. We then obtain the following generalization of Brody's formula:

$$p(n,s)=a_n s^{(n+1)q}\exp(-b_n s^{(n+1)q+1})\times\int_0^s p(n-1,x)\exp(b_n x^{(n+1)q+1})dx, \qquad (12)$$

for $n\geq 1$. Here $a_n=[(n+1)q+1]b_n$ in order to have a unit mean spacing, and b_n is defined by the normalization condition $\int_0^\infty p(n,s)ds=1$.

We shall now use the results of the numerical experiment by Engel *et al.* [14] on the hydrogen atom in a magnetic field and the Hénon-Heiles potential to test the ansatz (3). They applied Eq. (9) to the spacing distributions with $n=0,1,\ldots,7$ for both systems, taking q_n as a fitting parameter as we have mentioned above. We use Eq. (11) to calculate the chaoticity parameter q that corresponds to each of the eight values of q_n deduced in [14] for each state of the considered systems. We then find the mean value $\overline{q}$ and standard deviation σ_q, which correspond to each state. The results of calculation are given in Table I. In Fig. 1, we plot the ratio $q_n/(n+1)$ against n for values of q_n reported by Engel *et al.* for the hydrogen atom and Hénon-Heiles potential at different effective energies ε. The figure indicate that the best-fit values of q_n agree with Eq. (11), especially for the less chaotic systems. Table I shows that the ratio of the standard deviations of the values of q_n to the corresponding mean values for the systems under consideration are in the range from 16% to 1%, with a mean value of 7%. If we accept this

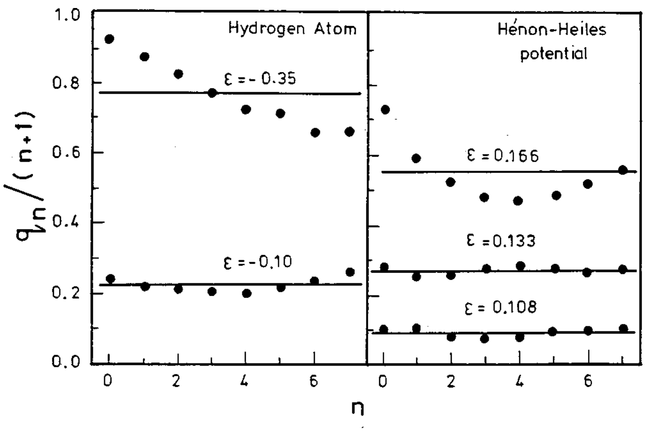

FIG. 1. The ratios of the best-fit parameters q_n of the nth-order spacing distributions obtained in [14] to the corresponding values of $(n+1)$, shown as experimental points, compared to the mean values $\overline{q}$ indicated by the horizontal lines.

ratio as an estimate for the agreement between the proposed power law and the analysis done by Engel *et al.*, we expect Eq. (12) to reproduce experimental high-order spacing distributions with a comparable accuracy. This accuracy is reasonable as long as the data are presented in the form of histograms. The agreement between the empirical values of q_n and Eq. (11) is, in our point of view, a further justification for the conclusion found in [13] that the degree of level repulsion is the major factor determining the spacing distributions of sufficiently high order ($n \sim 7$) and thus defines the correlation between the levels at distances several times exceeding the mean level spacing.

A generalization of the Brody formula for the NNNS distribution can be derived by substituting Eq. (8) for $p_B(0,s)$ into Eq. (12) to obtain

$$p_B(1,s) = a a_1 \int_0^s s^{2q} x^q \exp[-b x^{q+1} - b_1(s^{2q+1} - x^{2q+1})] dx. \tag{13}$$

The parameter a_1 is now equal to $(2q+1)b_1$, while $b_1(q)$ is obtained numerically from the normalization condition. We can have an exact result only in the case of a regular system where $b_1(0)=1$. In order to simplify the comparison with numerical experiments, we calculated the values of b_1 for different choices of q and parametrized the result in the form

$$b_1(q) = \frac{1}{1+2.7q+3.5q^2}. \tag{14}$$

Figure 2 demonstrates the behavior of the NNS and NNNS distributions for a system in a transition from regularity to chaos. We see from the figure that, while the two distributions show a smooth transition, the speed of evolution at different ranges of q is not the same. The modification introduced in the shape of the NNNS distribution by varying q from 0 to 0.33 is less than the corresponding modification of the NNS distribution, while both distributions ultimately reach the Poisson limit as $q=0$. This finding suggests that using the distributions (9) and (13) in a simultaneous analysis of the NNS and NNNS distributions can lead to an accurate determination of the power-repulsion exponent q.

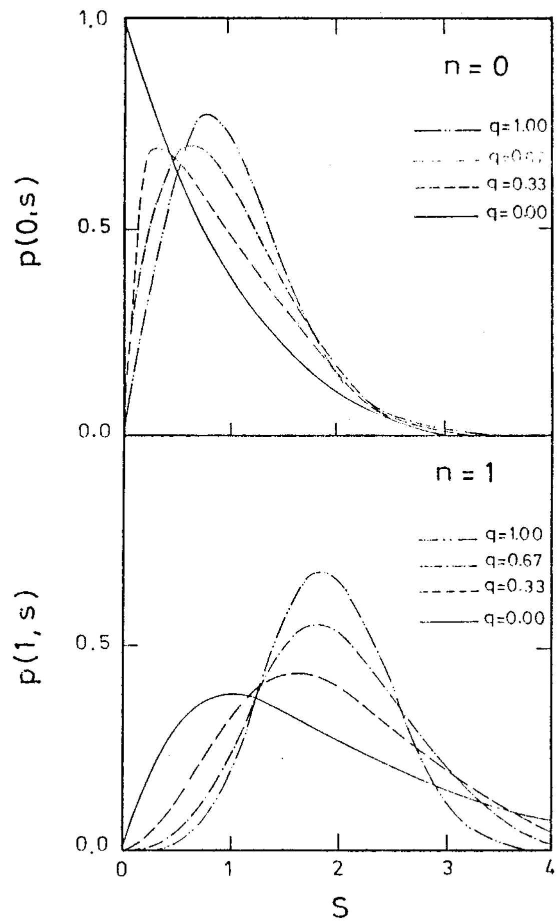

FIG. 2. NNS and NNNS distributions for mixed systems calculated using Brody's method for different level-repulsion exponents q.

III. SUPERPOSITION OF INDEPENDENT SPECTRA

This section considers the case of a composite spectrum resulting from a random superposition of N uncorrelated sequences of energy levels. This is the case when the system has a set of good quantum numbers such as spin and parity, which are not considered in the analysis of level statistics. In such a system, the Hamiltonian assumes block-diagonal form, and the total spectrum consists of a mixture of contributions from the individual subblocks [2]. The calculation of the NNS distribution of such a mixed spectrum is described as follows in Mehta's book [1]. Let ρ_i be the level density in the ith sequence and $P_i(x_i)$ be the NNS distribution for this sequence, with $x_i = f_i s$ and $f_i = \rho_i / \Sigma \rho_i$. The probability $E_i(x_i)$ that there is no level belonging to the ith sequence in a given interval of length x_i is related to the corresponding NNS distribution by

$$E_i(x_i) = \int_{x_i}^{\infty} F_i(x) dx = \int_{x_i}^{\infty} dx \int_x^{\infty} P_i(y). \tag{15}$$

Then the probability that a given interval of length s does not contain any of the levels of the mixed sequence is given by

$$E(s)=\prod_{i=1}^{N} E_i(x_i). \tag{16}$$

Differentiating Eq. (16) twice, one obtains

$$P(s)=E(s)\left\{\sum_{i=1}^{N} f_i^2 \frac{P_i(f_i s)}{E_i(f_i s)}+\left[\sum_{i=1}^{N} f_i \frac{F_i(f_i s)}{E_i(f_i s)}\right]^2 -\sum_{i=1}^{N}\left[f_i \frac{F_i(f_i s)}{E_i(f_i s)}\right]^2\right\}. \tag{17}$$

In order to obtain a similar expression for the nth-order spacing distribution, we consider the function $E(n,s)$ defined as the probability that an interval of length s of the spectrum contains exactly n levels. Because the individual sequences are assumed to be independent, the probability that a given interval s contains n levels of the mixed sequences is given by [16]

$$E(n,s)=\sum_{n_1,\ldots,n_N=0}^{n} \delta_{n_1+\cdots+n_N,n} \prod_{i=1}^{N} E_i(n_i,x_i). \tag{18}$$

We can check the validity of this relation considering the case when each of the individual sequences has a Poisson distribution (for each of the three functions E, F, and p) and showing that the resultant mixed system also has a Poisson distribution [16,17]. This function is related to the probability $F(n,s)$ that there are exactly n levels within a distance s from an arbitrary chosen level by [1]

$$-\frac{dE(n,s)}{ds}=F(n,s)-F(n-1,s). \tag{19}$$

The latter is related to the nth-order level spacing distribution $p(n,s)$ by

$$-\frac{dF(n,s)}{ds}=p(n,s)-p(n-1,s). \tag{20}$$

Differentiating Eq. (18) twice, and using Eqs. (19) and (20), we finally obtain

$$\frac{d^2E(n,s)}{ds^2}=\sum_{n_1,\ldots,n_N=0}^{n} \delta_{n_1+\cdots+n_N,n}\left\{\sum_{i=1}^{N} f_i^2 \frac{p_i(n_i,f_i s)+p_i(n_i-2,f_i s)-2p_i(n_i-1,f_i s)}{E_i(n_i,f_i s)} +\left[\sum_{i=1}^{N} f_i \frac{F_i(n_i,f_i s)-F_i(n_i-1,f_i s)}{E_i(n_i,f_i s)}\right]^2-\sum_{i=1}^{N}\left[f_i \frac{F_i(n_i,f_i s)-F_i(n_i-1,f_i s)}{E_i(n_i,f_i s)}\right]^2\right\}\prod_{i=1}^{N} E_i(n_i,f_i s). \tag{21}$$

This equation is valid for $n\geqslant 2$ but can be applied for the lower values if one defines $p(j,s)=F(j,s)=0$ for $j<0$. The nth-order level spacing distribution can now be obtained from Eq. (21) by using the following relation:

$$p(n,s)=\sum_{j=0}^{n} (n-j+1)\frac{d^2E(j,s)}{ds^2}, \tag{22}$$

which can easily be obtained by combining Eqs. (19) and (20).

An interesting special case of spectrum partition is when all the constituting sequences have GOE statistics. Such a situation occurs in a chaotic system when a symmetry is unknown or ignored [18]. In this case, each of the constituting sequences corresponds to a single eigenvalue (or set of eigenvalues) corresponding to the quantum number(s) of the unconsidered symmetry. A similar situation occurs in systems whose degrees of freedom can be divided into two noninteracting groups, one having chaotic dynamics and one regular [18]. This model has been able to reproduce the spacing distribution of low-lying levels of vibrational nuclei [19]. If all the N chaotic sequences have the same level density, then each sequence will have $f_i=1/N$ and $p_i(n,s)=p_W(n,s)$ given by Eqs. (4) and (5). In this case, Eq. (17) yields for the NNS distribution [18]

$$p_N(0,s)=\frac{1}{N}\left[\mathrm{erfc}\left(\frac{\sqrt{\pi}s}{2N}\right)\right]^N Q(s)\left[\frac{\pi s}{2N}+(N-1)Q(s)\right], \tag{23}$$

where

$$Q(s)=\frac{e^{-\pi s^2/4N^2}}{\mathrm{erfc}\left(\frac{\sqrt{\pi}s}{2N}\right)}. \tag{24}$$

Similarly, we use Eqs. (21) and (22) to obtain the following expression for the NNNS distribution:

$$p_N(1,s)=\frac{1}{N}\left[\mathrm{erfc}\left(\frac{\sqrt{\pi}s}{2N}\right)\right]^N\left[\frac{2\alpha^6 s^4}{N^4}R(s)+(N-1)Q(s) \times\{U_1(s)+U_2(s)\}\right], \tag{25}$$

where

$$U_1(s)=\left[(N-2)Q(s)+\frac{\pi s}{2N}\right] \times\left[\left(\frac{\alpha^2 s^2}{2N^2}+2\right)R(s)-\frac{s}{N}T(s)-2\right],$$

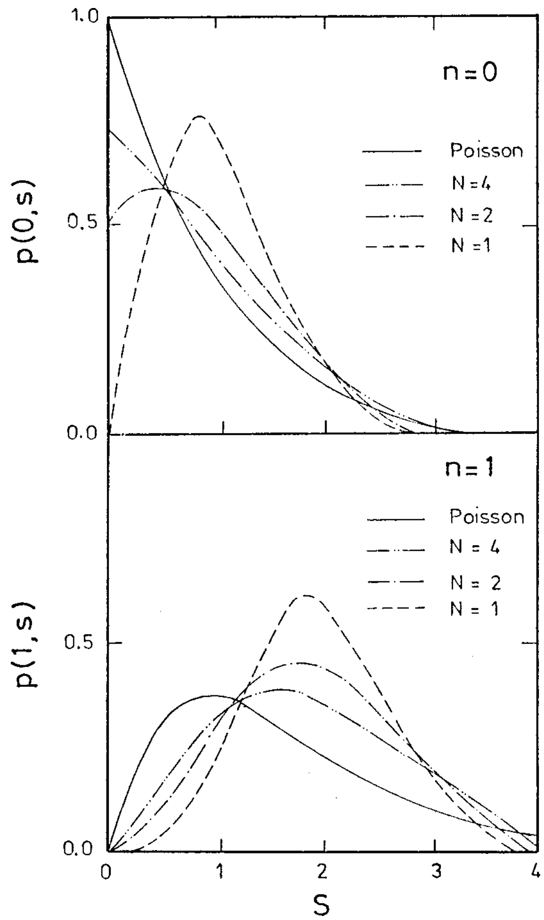

FIG. 3. NNS and NNNS distributions for spectra composed of random superpositions of N independent sequences.

$$U_2(s)=2\left[\frac{\alpha^2 s}{N}\left(\frac{\alpha^2 s^2}{N^2}+\frac{3}{2}\right)R(s)+T(s)-Q(s)\right], \quad (26)$$

$$R(s)=\frac{e^{-\alpha^2 s^2/N^2}}{\mathrm{erfc}\left(\dfrac{\sqrt{\pi}s}{2N}\right)}, \quad T(s)=\frac{\mathrm{erfc}(\alpha s/N)}{\mathrm{erfc}\left(\dfrac{\sqrt{\pi}s}{2N}\right)},$$

and $\alpha=4/3\sqrt{\pi}$. A similar though more complicated expression has been obtained for the second-NNS distribution.

Figure 3 shows the NNS and NNNS distributions calculated by means of Eqs. (23) and (25) for spectra consisting of N sequences, with $N=1$, 2, 4, and ∞. The latter two are evidently Poissonians. Although both the NNS and NNNS distributions ultimately reach the Poisson limit, the evolution of the NNNS distribution from the Wigner-type that characterizes the cases of $N=1$ is slower. This evolution is even smaller for the second-NNS distributions $p_N(2,s)$. This result together with the conclusion of the preceding section (Fig. 2) suggests that higher-order spacing distributions are more sensitive to the variation of the strength of perturbation in a nearly regular system than the NNS distribution, and is less sensitive in the end of the order-chaos transition.

IV. THE BERRY-ROBNIK METHOD

Berry and Robnik [5] proposed a semiclassical method for calculating the NNS distribution for systems with dynamics in the classical phase space where regular and irregular regions coexist on the energy surface. The method rests on the principle of uniform semiclassical condensation [20,16], which implies the semiclassical localization of the eigenfunctions either in the classical regular or classical irregular regions. The sequences of levels associated with these regions are assumed to be statistically independent, and their fraction densities f_i are determined by the invariant measure of the corresponding regions in classical phase space. The NNS distribution is then given by Eq. (17). The simplest and most important case occurs when the levels can be divided into two sequences, one for the regular motion and one for the chaotic with fractional level densities f_r and $f_c=1-f_r$, respectively. In this case, the gap function takes the following simple form:

$$E(0,s)=e^{-f_r s}\,\mathrm{erfc}\left(\frac{\sqrt{\pi}}{2}f_c s\right). \quad (27)$$

For the NNS distribution, Eq. (17) yields the following formula [5]:

$$p_{\mathrm{BR}}(0,s)=f_r^2 e^{-f_r s}\,\mathrm{erfc}\left(\frac{\sqrt{\pi}}{2}f_c s\right)+\left(2 f_r f_c+\frac{\pi}{2}f_c^3 s\right)e^{-f_r s-\pi f_c^2 s^2/4}, \quad (28)$$

which works well for systems in the deep semiclassical regime [10].

High-order gap functions have recently been considered by Prosen and Robnik [17]. They calculated the $E(n,s)$ statistics with different values of n being as large as 100 for three two-dimensional systems, namely the compactified standard map [10] with kick parameters equal to 1.8 and 0.04, and for a quartic billiard [21]. The result of the numerical calculation agrees very well, particularly for $n\leq 20$, with the theoretical calculation based on the principle of uniform semiclassical condensation, which yields in this case an equation of the type (18). Prosen and Robnik [17] applied a Poisson distribution for $E_{\mathrm{regular}}(n,s)$ for the regular components of the spectra. For the chaotic components, they used the numerical values given in Mehta's book [1] for $E_{\mathrm{chaotic}}(n,s)$ when $n\leq 7$ and the asymptotic formula suggested by Aurich *et al.* [22]. This agreement can be regarded as a confirmation of the Berry-Robnik picture for the $E(n,s)$ statistics.

For the NNNS distribution, we propose a generalization of the Berry-Robnik expression by our generalization of the Wigner surmise [13], which agrees perfectly well with Mehta's table for $E(n,s)$. We substitute Eqs. (1), (4), and (6) into Eqs. (21) and (22) to obtain

$$p_{\mathrm{BR}}(1,s)=(2-f_r s)\left[f_r f_c\,\mathrm{erfc}(\alpha f_c s)-f_r^2\,\mathrm{erfc}\left(\frac{\sqrt{\pi}}{2}f_c s\right)\right]\times e^{-f_r s}-f_r f_c\left(2-2f_r s-\frac{\pi}{2}f_c^2 s^2\right)\times e^{-f_r s-\pi f_c^2 s^2/4}+[2f_r^2+2\alpha^6 f_c^6 s^4+f_r f_c^2\alpha^2 s\times(3+\tfrac{1}{2}f_r s+2f_c^2\alpha^2 s^2)]e^{-f_r s-f_c^2\alpha^2/s^2}. \quad (29)$$

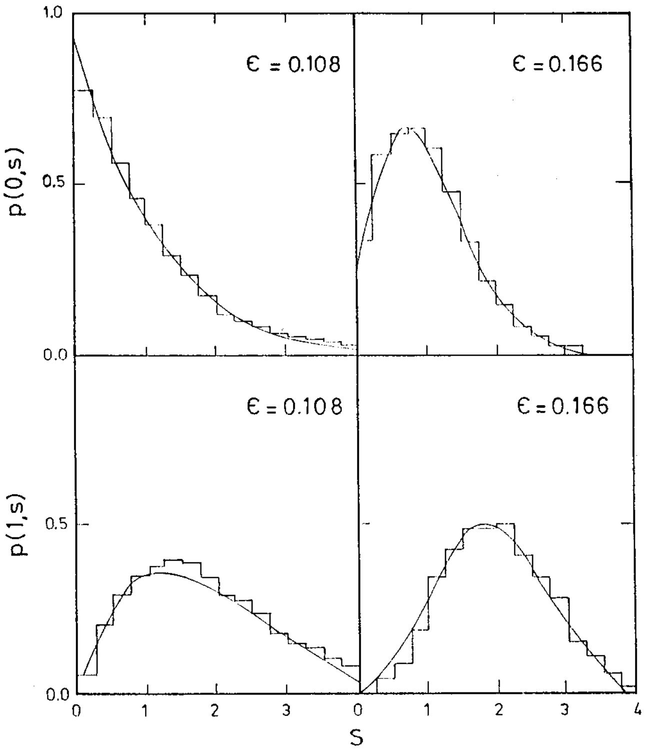

FIG. 4. NNS and NNNS distributions for mixed systems calculated using Berry and Robnik's method compared with the numerical experiment by Engel *et al.* [14] on the Hénon-Heiles potential with reduced energy ε.

Figure 4 shows the predictions of Eqs. (28) and (29) for the spacing distribution obtained in [14] for the levels of the Hénon-Heiles potential with reduced energies $\varepsilon=0.108$ and 0.166. We have used the NNS distributions to determine the best-fit values of the chaoticity parameter as $f_c=0.64$ in the former case and 0.13 in the latter. These values have been substituted into Eq. (29) to calculate the corresponding NNNS distributions. As the figure shows, Eq. (29) is very successful.

V. SUMMARY AND CONCLUSION

The study of the fluctuation properties of energy levels is the object of the random matrix theory. Short-range level correlations are usually expressed in terms of the NNS distribution $p(0,s)$. This distribution practically vanishes for spacings $s\gtrsim 3$, even in the case of a regular spectrum. Long-range level correlations are measured by other statistics. The most popular among those is the spectral rigidity $\Delta_3(L)$, which measures the mean-square deviation of the integrated level density from a straight line in an interval of length L [2]. The analysis of spectra in terms of the spectral rigidity is effective in the domain of $L\gtrsim 10$. The gap in information concerning the intermediate spacing can be filled by studying the higher-order spacing distributions $p(n,s)$, with $n\geqslant 1$. In Ref. [13], we propose a generalization of the Wigner surmise that describes the spacing distributions of any order for a chaotic system. The present paper considers the spacing distributions $p(n,s)$ for systems with mixed regular-chaotic dynamics. We apply methods similar to those, which have been used to derive the Brody and Berry-Robnik formulas for the NNS distribution for the levels of a mixed system. The former method has already been applied by Engel *et al.* [14], and we propose here a slight modification of that work. These authors took the power of level repulsion of each order as a fitting parameter, but we did not. We used instead the relation (11) between the level-repulsion exponents. Our results agree with theirs to within 7%. We then obtained a general expression for $p(n,s)$, which describes the case in which the spectrum consists of a superposition of independent sequences. This applies when a good quantum number is ignored in the spectral analysis of a chaotic system. This expression is used to obtain a generalization of the Berry-Robnik formula for the NNNS distribution, which successfully describes mixed systems in the deep semiclassical regime. The obtained formula is tested successfully by following the transition from regularity to chaos of the motion of a particle in the Hénon-Heiles potential, which confirms the validity of the principle of uniform semiclassical condensation for high-order spacing distributions. This finding agrees with the recent results obtained by Prosen and Robnik [17] concerning the nth-order gap function $E(n,s)$, which also confirms the validity of the Berry-Robnik picture for high-order spacing distributions.

We have also shown that when the tuning parameter that describes the degree of disorder is varied starting from the value that implies full chaos, the NNNS distribution $p(1,s)$ departs from the Wigner-like shape slower than the corresponding deviation of the NNS distribution $p(0,s)$, while both distributions ultimately reach the Poissonian limit when the system becomes fully regular. This finding suggests using both distributions simultaneously in the analysis of level statistics of a mixed system. We expect $p(0,s)$ to give an accurate measure of the chaoticity parameter when the system is in its early stages of transition from regularity to chaos, while $p(1,s)$ is expected to be more useful in the later stages.

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